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# FINITE GROUPS WITH DENSE $\mathcal{CD}$-SUBGROUPS

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ABSTRACT. A group $G$ is said to have dense $\mathcal{CD}$-subgroups if each non-empty open interval of the subgroup lattice $L(G)$ contains a subgroup in the Chermak–Delgado lattice $\mathcal{CD}(G)$. In this note, we study finite groups satisfying this property.

## 1. Introduction

Let $\mathcal{X}$ be a property pertaining to subgroups of a group. We say that a group $G$ has *dense $\mathcal{X}$-subgroups* if for each pair $(H, K)$ of subgroups of $G$ such that $H < K$ and $H$ is not maximal in $K$, there exists a $\mathcal{X}$-subgroup $X$ of $G$ such that $H < X < K$. Groups with dense $\mathcal{X}$-subgroups have been studied for many properties $\mathcal{X}$: being a normal subgroup [9], a pronormal subgroup [19], a normal-by-finite subgroup [5], a nearly normal subgroup [6] or a subnormal subgroup [7].

In our note, we will focus on the property of being a $\mathcal{CD}$-subgroup, that is a subgroup contained in the Chermak–Delgado lattice. Let $G$ be a finite group and $L(G)$ be the subgroup lattice of $G$. We recall that the *Chermak–Delgado measure* of a subgroup $H$ of $G$ is defined by

$$m_G(H) = |H||C_G(H)|.$$

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Let

$$m^*(G) = \max\{m_G(H) \mid H \leq G\} \text{ and } \mathcal{CD}(G) = \{H \leq G \mid m_G(H) = m^*(G)\}.$$

Then the set $\mathcal{CD}(G)$ forms a modular, self-dual sublattice of $L(G)$, which is called the *Chermak–Delgado lattice* of $G$. It was first introduced by Chermak and Delgado [4], and revisited by Isaacs [8]. In the last years there has been a growing interest in understanding this lattice (see e.g. [1]-[3], [10]-[14], [16]-[18] and [20]). We also recall several important properties of the Chermak–Delgado measure:

- if $H \leq G$ then $m_G(H) \leq m_G(C_G(H))$, and if the measures are equal, then $C_G(C_G(H)) = H$;
- if $H \in \mathcal{CD}(G)$ then $C_G(H) \in \mathcal{CD}(G)$ and $C_G(C_G(H)) = H$;
- the maximal member $M$ of $\mathcal{CD}(G)$ is characteristic and satisfies $\mathcal{CD}(M) = \mathcal{CD}(G)$;
- the minimal member $M(G)$ of $\mathcal{CD}(G)$ (called the *Chermak–Delgado subgroup* of $G$) is characteristic, abelian and contains $Z(G)$.

Our main results are stated as follows. The first one gives some information about finite $p$-groups having dense $\mathcal{CD}$-subgroups. Obviously, finite $p$-groups of order $p$ satisfy this property.

**Theorem 1.1.** *Let $G$ be a finite $p$-group of order $p^n$, $n \geq 2$, having dense $\mathcal{CD}$-subgroups. Then:*

a) $|Z(G)| = p$ *and* $m^*(G) = p^{n+1}$.
b) *All subgroups of order $p^2$ of $G$ contain $Z(G)$. Moreover, all normal subgroups of order $p^2$ of $G$ are contained in $\mathcal{CD}(G)$.*
c) $\mathrm{Im}(m_G) = \{p^n, p^{n+1}\}$.

Theorem 1.1 shows that the finite $p$-groups with dense $\mathcal{CD}$-subgroups are a subclass of the class $\mathcal{C}$ described in Section 3 of [17]. Note that all extraspecial $p$-groups belong to class $\mathcal{C}$, but only some of the extraspecial $p$-groups have dense $\mathcal{CD}$-subgroups. For $p$ odd, all extraspecial groups of order $p^3$ have dense $\mathcal{CD}$-subgroups, and all extraspecial groups of order $p^k$, $k \geq 5$, do not have dense $\mathcal{CD}$-subgroups (such groups have subgroups of order $p^2$ that intersect the center trivially). For $p = 2$ the situation is similar, except that one of the extraspecial groups of order 32 has dense $\mathcal{CD}$-subgroups and the other one does not.

The second result classifies finite groups which are not $p$-groups, but have dense $\mathcal{CD}$-subgroups.

**Theorem 1.2.** *Let $G$ be a finite group with $|\pi(G)| \geq 2$. The group $G$ has dense $\mathcal{CD}$-subgroups if and only if $G$ is a nonabelian group of order $pq$, where $p$ and $q$ are primes.*

Most of our notation is standard and will not be repeated here. Basic definitions and results on groups can be found in [8]. For subgroup lattice concepts we refer the reader to [15].

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## 2. **Proofs of the main results**

*Proof of Theorem 1.1.* a) Let $H \leq G$ with $|H| = p^2$. By hypothesis, there is $K \in \mathcal{CD}(G)$ such that $1 < K < H$. Then $|K| = p$ and since $Z(G) \subseteq K$, it follows that $Z(G) = K$. Thus $|Z(G)| = p$, $Z(G) \in \mathcal{CD}(G)$ and $m^*(G) = p^{n+1}$.

b) Since every subgroup of order $p^2$ of $G$ contains a $\mathcal{CD}$-subgroup of order $p$, we infer that it contains $Z(G)$. The second statement follows from Corollary 4.5.5 of [18].

c) Suppose $H \leq G$ with $|H| = p$ and $H \neq Z(G)$. Then there is $K \in \mathcal{CD}(G)$ such that $|K| = p^2$ and $H \subset K$. We get $C_G(K) \subseteq C_G(H)$ and since $C_G(K)$ is of order $p^{n-1}$, we obtain $C_G(H) = C_G(K)$. Thus $m_G(H) = |H||C_G(H)| = p \cdot p^{n-1} = p^n$.

Now, suppose $A \leq G$ with $|A| \geq p^2$ and $A \notin \mathcal{CD}(G)$. Then there are $B, C \in \mathcal{CD}(G)$ such that $B \subset A \subset C$ and $|A|/|B| = |C|/|A| = p$. It follows that $C_G(C) \subseteq C_G(A) \subseteq C_G(B)$ and $|C_G(B)|/|C_G(C)| = p^2$. We cannot have $C_G(A) = C_G(B)$ because this implies $m_G(A) > m^*(G)$, a contradiction. Also, we cannot have $|C_G(A)| = p|C_G(C)|$ because this implies $A \in \mathcal{CD}(G)$, a contradiction. So, $C_G(A) = C_G(C)$, which leads to $m_G(A) = p^n$. Finally, note that $m_G(1) = |G| = p^n$. Thus $\mathrm{Im}(m_G) = \{p^n, p^{n+1}\}$.

□

*Proof of Theorem 1.2.* If $G$ is a nonabelian group of order $pq$, $p$ and $q$ primes, then $1 < N < G$ where $N$ is a normal subgroup of prime order, and $\mathcal{CD}(G) = \{N\}$, and thus $G$ has dense $\mathcal{CD}$-subgroups.

Conversely, suppose that $G$ has dense $\mathcal{CD}$-subgroups. Let $|G| = p_1^{n_1} \cdots p_k^{n_k}$ with $k \geq 2$. If $n_i = 1$, $\forall\, i = 1, \ldots, k$, then $G$ is a ZM-group and so its Chermak–Delgado lattice is a chain of length 0 by Theorem 3 of [12]. This satisfies the density property if and only if $k = 2$, that is, $G$ is a nonabelian group of order $p_1 p_2$.

In what follows, we assume that $n_1 \geq 2$ and let $H_1 \leq G$ with $|H_1| = p_1^2$. Then between 1 and $H_1$ there is a subgroup $K_1 \in \mathcal{CD}(G)$ of order $p_1$. Since $Z(G) \subseteq K_1$, we have the following two cases:

**Case 1.** $Z(G) = K_1$.
Then $n_2 = \cdots = n_k = 1$ and so $|G| = p_1^{n_1} p_2 \cdots p_k$. Indeed, if $n_i \geq 2$ for some $i = 2, \ldots, k$, then there is $K_i \in \mathcal{CD}(G)$ of order $p_i$. This implies $Z(G) \subseteq K_i$, that is $K_1 \subseteq K_i$, which leads to $p_1 \mid p_i$, a contradiction.

Clearly, we can choose $M_1 \leq G$ with $|M_1| = p_1^{n_1 - 1}$ and $M_1 \in \mathcal{CD}(G)$. Now, subgroups in $\mathcal{CD}(G)$ are permutable, and $\mathcal{CD}(G)$ is closed under conjugation, and so the normal closure $M_1^G$ is a $p_1$-subgroup. Now the quotient group $G/M_1^G$ is a ZM-group and hence solvable, and $M_1^G$ is a $p_1$-subgroup which is solvable, and so it follows that $G$ is solvable. (Note: the solvability of a ZM-group can be proven as a consequence of Burnside's normal complement theorem. Even more is true: in fact a ZM-group is metacyclic.)

We observe that $k = 2$. Indeed, if $k \geq 3$, then as $G$ is solvable it contains a Hall subgroup $H_{23}$ of order $p_2p_3$. Let $K \leq G$ such that $1 \subset K \subset H_{23}$ and $K \in \mathcal{CD}(G)$. Then $|K| \in \{p_2, p_3\}$ and $Z(G) \subseteq K$, implying that either $p_1 \mid p_2$ or $p_1 \mid p_3$, a contradiction. Thus

$$|G| = p_1^{n_1}p_2 \text{ and } m^*(G) = p_1^{n_1+1}p_2.$$

So $|C_G(M_1)| = p_1^2p_2$ and let $P_2$ be a Sylow $p_2$-subgroup of $G$ that centralizes $M_1$. Now, the interval $(M_1, G)$ of $L(G)$ contains $\mathcal{CD}$-subgroups and such a subgroup is either a Sylow $p_1$-subgroup $P_1$ of $G$ or a product $M_1P_2^y$ for some $y \in G$. If $P_1 \in \mathcal{CD}(G)$, we get $|C_G(P_1)| = p_1p_2$ and therefore there is $x \in G$ such that $P_2^x$ centralizes $P_1$. Since $P_2^x$ centralizes $P_2^x$, it follows that $P_2^x$ centralizes $P_1P_2^x = G$. So, $P_2^x \subseteq Z(G)$, implying that $p_2 \mid p_1$, a contradiction. We now argue that $M_1$ is normal in $G$. Otherwise, the normal closure $M_1^G$ is a Sylow $p_1$-subgroup of $G$ that is in $\mathcal{CD}(G)$, a contradiction. So $M_1$ is normal in $G$ and suppose $M_1P_2^y \in \mathcal{CD}(G)$ for some $y \in G$. Then $M_1P_2 \in \mathcal{CD}(G)$ and so $|C_G(M_1P_2)| = p_1^2$. But $P_2$ centralizes $M_1P_2$ and consequently $p_2 \mid p_1^2$, a contradiction.

Thus this case is impossible.

**Case 2.** $Z(G) = 1$.
Since $M(G) \subseteq K_1$, we have either $M(G) = 1$ or $M(G) = K_1$.

If $M(G) = 1$ then $1 \in \mathcal{CD}(G)$ and so $\mathcal{CD}(G)$ does not contain any subgroup of prime order by Corollary 7 of [11]. This contradicts our hypothesis.

Assume now that $M(G) = K_1$. Similarly with Case 1, we get $n_2 = \cdots = n_k = 1$ and $k = 2$, that is $|G| = p_1^{n_1}p_2$. Since $|C_G(K_1)|$ is a proper divisor of $|G|$ and $p_1|C_G(K_1)| = m_G(K_1) > m_G(1) = |G|$, it follows that $|C_G(K_1)| = p_1^{n_1}$ and hence $m^*(G) = p_1^{n_1+1}$. Let $P_2$ be a Sylow $p_2$-subgroup of $G$. Since $|G| = p_1^{n_1}p_2$ and $n_1 \geq 2$, we have $P_2 < K_1P_2 < G$, and so $P_2$ is not maximal in $G$. Thus, there is $X \in \mathcal{CD}(G)$ so that $P_2 < X < G$, and hence $p_2 \mid |X|$, contradicting $m^*(G) = p_1^{n_1+1}$. □

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